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1999 J. Phys. A: Math. Gen. 32 6361

(http://iopscience.iop.org/0305-4470/32/36/301)

J. Phys. A: Math. Gen. **32** (1999) 6361–6371. Printed in the UK PII: S0305-4470(99)04133-5

High-precision Monte Carlo study of the 3D XY-universality class

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Received 6 May 1999

Abstract. We present a Monte Carlo study of the two-component ϕ^4 model on the simple cubic lattice in three dimensions. By suitable tuning of the coupling constant, λ, we eliminate leading-order corrections to scaling. High-statistics simulations using finite-size scaling techniques yield $\nu = 0.6723(3)[8]$ and $\eta = 0.0381(2)[2]$, where the statistical and systematical errors are given in the first and second bracket, respectively. These results are more precise than any previous theoretical estimate of the critical exponents for the 3D XY universality class.

1. Introduction

The 3D XY universality class is unique in the respect that experimental estimates for critical exponents are more precise than any theoretical estimate. These experiments are performed in the neighbourhood of the super-fluid transition of ^{4}He. The specific heat or the super-fluid density is measured as a function of the temperature [1–3].

In the present study we try to close the gap between theory and experiment by a high-statistics Monte Carlo simulation of the two-component ϕ^4 (or Landau–Ginzburg) model on a 3D simple cubic lattice. The action is given by

$$S = \sum_x \left\{ -2\kappa \sum_\mu \vec{\phi}_x \cdot \vec{\phi}_{x+\hat{\mu}} + \vec{\phi}_x^2 + \lambda(\vec{\phi}_x^2 - 1)^2 \right\} \tag{1}$$

where the field variable, $\vec{\phi}_x$, is a vector with two real components and $x = (x_1, x_2, x_3)$, where x_i is an integer, labels the lattice sites. μ labels the directions and $\hat{\mu}$ is a unit vector in the μ-direction. The Boltzmann factor is $\exp(-S)$. For $\lambda = 0$ we get the Gaussian model on the lattice. In the limit $\lambda = \infty$ the XY model is recovered.

In addition to statistical errors Monte Carlo estimates of critical exponents are affected by systematical errors that result from corrections to scaling. These systematical errors can be reduced (in a finite-size scaling study) by increasing the linear size, L, of the lattices that are simulated. A more elegant approach is to remove corrections by a suitable choice of the action. Recently, it was demonstrated that leading-order corrections to scaling can be removed by a suitable tuning of the coupling constant λ in the one-component ϕ^4 theory on the lattice [4–6]. Leading-order corrections to scaling are proportional to $\xi^{-\omega}$ ($L^{-\omega}$ in finite-size scaling), where ξ is the correlation length and $\omega \approx 0.8$.

The paper is organized as follows. In section 2 we discuss the observables that are measured. In section 3 we explain the algorithm that has been used for the simulation and

0305-4470/99/366361+11$30.00

we summarize the simulation parameters. In section 4 the data are analysed. In section 5 our results for exponents are compared with experimental and theoretical estimates given in the literature. In section 6 we give our conclusions and an outlook.

2. The measured quantities

In the case of the one-component model the Binder cumulant turned out to be a good indicator for corrections to scaling [6]. The Binder cumulant is defined by

$$U = \frac{\langle(\vec{m}^2)^2\rangle}{\langle\vec{m}^2\rangle^2} \tag{2}$$

where

$$\vec{m} = \frac{1}{V}\sum_x \vec{\phi}_x \tag{3}$$

is the magnetization per lattice site of a given configuration. The volume is $V = L^3$. In the following we always consider systems with periodic boundary conditions. In [6] the Binder cumulant was computed at a fixed value of the ratio of partition functions Z_a/Z_p. Z_a is the partition function for anti-periodic boundary conditions and Z_p for periodic boundary conditions. This ratio can also be computed for an arbitrary number of components. For a simulation of the XY model see [7]. However, in the present paper we have replaced Z_a/Z_p by the dimensionless ratio ξ_{2nd}/L, because the second-moment correlation length, ξ_{2nd}, is easier to implement as Z_a/Z_p. Note that ξ/L, where ξ is the exponential correlation length on a strip of width L, was used in the pioneering work of Nightingale [8] on the phenomenological renormalization group approach.

The second-moment correlation length is defined by

$$\xi_{2nd} = \left(\frac{\chi/F - 1}{4\sin^2(\pi/L)}\right)^{1/2} \tag{4}$$

where the magnetic susceptibility is given by

$$\chi = V\langle\vec{m}^2\rangle \tag{5}$$

and

$$F = \frac{1}{V}\left\langle\left|\sum_x \exp\left(\mathrm{i}\frac{2\pi x_1}{L}\right)\vec{\phi}_x\right|^2\right\rangle \tag{6}$$

is the Fourier transform of the correlation function at minimal momentum. In the simulation we averaged over all three directions to reduce the statistical error. Note that in the following ξ_{2nd} is always evaluated at a finite value of L and not for the thermodynamic limit.

We performed some simulations of the one-component model to compare ξ_{2nd}/L and Z_a/Z_p. We found that the physical as well as statistical properties of ξ_{2nd}/L and Z_a/Z_p are similar.

In order to compute observables in the neighbourhood of the simulation parameter, κ_s, we computed the first two coefficients of the Taylor expansion in $\kappa - \kappa_s$. We always checked that the errors made by the truncation of the Taylor series are much smaller than the statistical errors of the quantities that were computed.

3. The simulations

3.1. The Monte Carlo algorithm

We generalize the idea of Brower and Tamayo [9] to simulate the one-component ϕ^4 theory. They used the Swendsen–Wang cluster algorithm [10] to update the sign of the field ϕ. In order to obtain an ergodic update they supplement the cluster update with a Metropolis update that also allows one to update the modulus of the field. In our case we only use the single-cluster algorithm [11] to update the direction of the field. The modulus is updated with the Metropolis algorithm. Let us briefly recall the steps of the single-cluster algorithm applied to the two-component ϕ^4 theory. First, a direction $\vec{n}$ is chosen

$$n_1 = \sin(2\pi\theta) \qquad n_2 = \cos(2\pi\theta) \tag{7}$$

where θ is a random number that is uniformly distributed in $[0, 1)$. Next a site of the lattice is picked randomly as the seed of the cluster. The cluster is built recursively. New sites enter the cluster when they freeze onto their neighbours that are already members of the cluster. The freezing probability is $p_f = 1 - p_d$ with

$$p_d = \min[1, \exp(-4\kappa(\vec{n} \cdot \vec{\phi}_x)(\vec{n} \cdot \vec{\phi}_y))]. \tag{8}$$

The fields of all sites in the cluster are reflected

$$\vec{\phi}'_x = \vec{\phi}_x - 2(\vec{n} \cdot \vec{\phi}_x)\vec{n}. \tag{9}$$

The modulus of $\vec{\phi}$ is changed with a local Metropolis update. A proposal for the field is generated by

$$\phi'_{i,x} = \phi_{i,x} + s(r_i - 0.5) \tag{10}$$

for $i = 1, 2$, where r_i is a random number that is uniformly distributed in $[0, 1)$. The acceptance probability is given by

$$A = \min[1, \exp(S - S')] \tag{11}$$

where S and S' are the action for the original field and the proposal, respectively. We found that a step-size $s = 2$ yields an acceptance rate of about 50%. In one sweep we go through the lattice in lexicographic order.

3.2. The simulation parameters

The program is written in `C`. As a pseudo random number generator we used our own implementation of `G05CAF` of the NAG-library. As a test of the correctness of the program and of the quality of the random number generator we compared Monte Carlo results for $\lambda = 0$ with exact results and Monte Carlo results for small β with high-temperature series expansions [12]. The program and the random number generator passed these tests. Note that linear congruential pseudo-random number generators with periods much smaller than that of `G05CAF` have passed tests that apply the single cluster and Metropolis algorithm to the 2D and 3D Ising model (see e.g. [13]).

We performed simulations at a large range of λ values and linear lattice sizes L. In table 1 we give an overview of the simulation parameters and the number of measurements for each set of simulation parameters. Most of our simulations were performed on 200 MHz Pentium Pro PCs running under `Linux`.

For each measurement we performed one sweep with the Metropolis algorithm and m single cluster updates. The number of cluster updates was chosen as roughly proportional to

Table 1. Summary of simulation parameters. In the first column we give the value of λ, in the second column the linear lattice size L and in the third column the number of measurements divided by 3×10^6.

λ	L	Stat/3×10^6
0.5	8, 16	17, 5
1.0	6, 8, 10, 12, 14, 16, 18, 20, 22, 24	50, 10, 10, 10, 10, 10, 10, 10, 10, 11
1.5	8, 16	13, 6
1.7	8, 12, 24	15, 10, 2.5
1.8	3, 4, 5, 6, 7, 8, 9, 10, 12, 16	20, 67, 67, 20, 40, 15, 45, 30, 15, 8
1.9	3, 4, 5, 6, 7, 8, 12, 16, 20, 24	33, 27, 20, 20, 15, 20, 10, 10, 11, 10
1.98	8, 12, 16, 20, 24	20, 15, 10, 11, 15
2.0	3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	133, 67, 67, 50, 25, 20, 24, 20, 20, 20, 20, 30
	15, 16, 18, 20, 22, 24, 26, 28, 32, 40, 48	20, 20, 25, 25, 25, 20, 16, 15, 22, 10, 10
2.2	3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 24	133, 67, 67, 50, 40, 15, 45, 30, 15, 8, 5
4.0	6, 7, 8, 9, 10, 11, 12, 14, 16, 18, 20, 22, 24	50, 20, 10, 10, 10, 9, 10, 10, 10, 10, 10, 10, 10

the linear lattice size L. For some lattice sizes we searched for the m value that gives the optimal performance of the algorithm. For $L = 48$ we found $m = 40$ as optimal.

The total amount of CPU-time used for the simulations was about 3 years on the 200 MHz Pentium Pro PCs.

4. Analysing the data

4.1. The Binder cumulant and corrections to scaling

We analysed the Binder cumulant at $\xi_{2nd}/L = 0.5927$ fixed. This means that first (at fixed λ) κ_f is computed for that $\xi_{2nd}/L = 0.5927$. Then, the Binder cumulant is computed at κ_f. In the following we denote the Binder cumulant at $\xi_{2nd}/L = 0.5927$ by $\bar{U}$. From preliminary simulations we know that $\xi_{2nd}/L = 0.5927$ is a good approximation of

$$\xi_{2nd}/L^* = \lim_{L\to\infty} \xi_{2nd}/L|_{\kappa_c}. \tag{12}$$

The advantage of this approach is that we need not search for κ_c and that, due to cross correlations, the statistical error of $\bar{U}$ is smaller than that of $U|_{\kappa_c}$ (see e.g. [14]).

For large L, $\bar{U}$ approaches a universal constant $\bar{U}^*$. Leading-order corrections are given by

$$\bar{U}(L,\lambda) = \bar{U}^* + c_1(\lambda)L^{-\omega}. \tag{13}$$

We fitted the data for all values of λ simultaneously with this ansatz. The free parameters of this fit are $\bar{U}^*$, ω and $c_1(\lambda)$ for each value of λ.

The results for various minimal lattice sizes L_{min} that have been included in the fit are summarized in table 2. The values for χ^2/d.o.f. stay rather large as L_{min} is increased. We could not pinpoint the particular problem that caused this effect. On the other hand, the result for the exponent ω is quite stable as L_{min} is varied. As our final result for the correction to scaling exponent we quote $\omega = 0.79(2)$. It is hard to give reliable estimates for the systematical errors. The fact that the result for ω stays almost constant starting from $L_{min} = 6$ at least indicates that these errors should be small.

For $L_{min} = 12$, 14 and 16 we give the results for $c_1(\lambda)$ in table 3. Linear interpolation of the result for c_1 at $\lambda = 2.0$ and 2.2 yields $\lambda_{opt} = 2.046(9)$, 2.086(9) and 2.101(10) for $L_{min} = 12$, 14 and 16, respectively. Where λ_{opt} is defined by $c(\lambda_{opt}) = 0$. There is still an

Table 2. Fit results for the Binder cumulant evaluated at $\xi_{2nd}/L = 0.5927$ fixed. The ansatz is given in equation (13). We give results for various minimal lattice sizes L_{min}, ω is the correction to the scaling exponent.

L_{min}	χ^2/d.o.f.	$\bar{U}^*$	ω
6	5.42	1.243 57(3)	0.786(6)
8	2.34	1.243 24(4)	0.775(6)
10	2.15	1.243 11(5)	0.788(10)
12	1.80	1.242 97(6)	0.782(14)
14	1.75	1.242 79(8)	0.790(20)
16	1.86	1.242 74(9)	0.819(31)

Table 3. The correction to scaling amplitude $c_1(\lambda)$ as a function of λ from fits with ansatz (13). We give the results for three values of $L_{min} = 12$, 14 and 16.

	L_{min}		
λ	12	14	16
0.5	0.2152(83)	0.2220(122)	0.2408(207)
1.0	0.0956(37)	0.0999(59)	0.1094(101)
1.5	0.0398(21)	0.0424(28)	0.0464(43)
1.7	0.0229(12)	0.0280(32)	0.0314(42)
1.8	0.0153(8)	0.0186(17)	0.0207(23)
1.9	0.0077(8)	0.0099(11)	0.0114(15)
1.98	0.0038(7)	0.0067(11)	0.0079(14)
2.0	0.0022(6)	0.0043(9)	0.0057(13)
2.2	−0.0074(7)	−0.0057(13)	−0.0056(16)
4.0	−0.0604(24)	−0.0601(37)	−0.0649(63)

increase in λ_{opt} visible as L_{min} increases. We quote $\lambda_{opt} = 2.10(1)[5]$ as our final result. As a rough estimate of systematical errors we give (in the square brackets) the difference of the result for $L_{min} = 12$ and 16.

Following [6] we tried to fit our data with the extended ansatz

$$\bar{U}(L,\lambda) = \bar{U}^* + c_1(\lambda)L^{-\omega} + c_2 c_1(\lambda)^2 L^{-2\omega}. \tag{14}$$

However, it turned out that we had too few data with a large enough difference, $\bar{U} - \bar{U}^*$, to resolve c_2.

Finally, we fitted the difference of the Binder cumulant at $\lambda = 2.0$ and 2.2 with the ansatz

$$\bar{U}(L,\lambda = 2.0) - \bar{U}(L,\lambda = 2.2) = cL^{-\omega}. \tag{15}$$

The results are given in table 4. It turns out that χ^2/d.o.f. is already of the order of one for the very small $L_{min} = 3$. Also, the value obtained for ω with this small L_{min} is consistent with the result obtained above. Hence, corrections beyond $L^{-\omega}$ depend very little on λ and are cancelled in $\bar{U}(L,\lambda = 2.0) - \bar{U}(L,\lambda = 2.2)$. The same observation holds in the case of the one-component model [6].

4.2. The critical line $\kappa_c(\lambda)$

As an approximation of the critical κ_c we take κ_f where $\xi_{2nd}/L = 0.5927$. In table 5 we give the result for the largest lattice size available for each value of λ that has been studied. Leading corrections are given by

$$\kappa_f - \kappa_c = aL^{-1/\nu} + bL^{-1/\nu-\omega} + \cdots. \tag{16}$$

Table 4. Fitting the difference of $\bar{U}$ at $\lambda = 2.0$ and 2.2 with ansatz (15).

L_{min}	ω	$c_1(2.0) - c_2(2.2)$	χ^2/d.o.f.
3	0.787(18)	0.0106(3)	0.98
4	0.780(31)	0.0104(5)	1.09
5	0.794(43)	0.0107(9)	1.21

Table 5. Estimates of the critical κ_c for all values of λ that have been simulated. The value for $\lambda = \infty$ has been taken from [14]. The systematical errors are given in square brackets and statistical errors in round brackets.

λ	L	$2\,\kappa_c$
0		0.33...
0.5	16	0.482 8[6]
1.0	24	0.507 54[7]
1.5	16	0.511 97[7]
1.7	24	0.511 60[2]
1.8	16	0.511 15[2]
1.9	24	0.510 576(2)[7]
1.98	24	0.510 049(1)[7]
2.0	48	0.509 9049(6)[9]
2.2	24	0.508 344(2)[4]
4.0	24	0.492 43[5]
∞		0.454 165(4)

Table 6. Fits of the magnetic susceptibility at $\xi_{2nd}/L = 0.5927$ fixed with ansatz (17).

L_{min}	d	η	χ^2/d.o.f.
14	1.256 29(20)	0.036 67(5)	10.28
24	1.259 57(50)	0.037 42(11)	2.91
26	1.260 67(61)	0.037 66(14)	0.70
28	1.261 17(75)	0.037 77(17)	0.40

The constant a should be very small since we have chosen $\xi_{2nd}/L = 0.5927$ as a good approximation of ξ_{2nd}/L^*. The value of b depends on λ and vanishes at λ_{opt}. Nevertheless, we pessimistically assume that errors decay with $L^{-1/\nu}$. Systematical errors are then computed by comparing κ_f at L with κ_f at $L/2$. These errors are given in square brackets. Whenever statistical errors reach a similar size as the systematical ones they are quoted in addition, in round brackets.

4.3. The exponent η

We computed the exponent η from the finite-size behaviour of the magnetic susceptibility, χ, at either $\xi_{2nd}/L = 0.5927$ or $U = 1.243$ fixed. We denote the magnetic susceptibility at ξ_{2nd}/L or U fixed by $\bar{\chi}$. It scales as

$$\bar{\chi} = \mathrm{d}L^{2-\eta}. \tag{17}$$

First we analysed our data for $\lambda = 2.0$ which is close to λ_{opt} and where we have accumulated the most data. Results for fixed ξ_{2nd}/L are given in table 6 and for fixed U in table 7.

In both cases rather large L_{min} are needed to reach an χ^2/d.o.f. value close to one. Since $\bar{\chi}$ at fixed ξ_{2nd}/L has a smaller statistical error than $\bar{\chi}$ at fixed U, the statistical error of η is

Table 7. Fits of the magnetic susceptibility at $U = 1.243$ fixed with ansatz (17).

L_{min}	d	η	χ^2/ d.o.f.
14	1.2598(6)	0.037 40(16)	2.27
22	1.2628(13)	0.038 11(31)	1.20
24	1.2644(16)	0.038 45(38)	0.83
26	1.2625(20)	0.038 04(46)	0.29

Table 8. Fits of the magnetic susceptibility at $\xi_{2nd}/L = 0.5927$ fixed with the extended ansatz (18).

L_{min}	c	d	η	χ^2/d.o.f.
6	−0.3602(40)	1.261 87(21)	0.037 84(5)	2.07
7	−0.3809(68)	1.262 46(26)	0.037 98(6)	1.33
8	−0.395(11)	1.262 80(34)	0.038 05(8)	1.17
10	−0.381(18)	1.262 54(43)	0.038 00(10)	1.06
12	−0.393(32)	1.262 75(62)	0.038 04(14)	1.21
14	−0.405(43)	1.262 89(73)	0.038 07(16)	1.40
16	−0.436(72)	1.263 30(99)	0.038 15(21)	1.32

Table 9. Fits of the magnetic susceptibility at $U = 1.243$ fixed with the extended ansatz (18).

L_{min}	c	d	η	χ^2/d.o.f.
4	−0.464(4)	1.2651(4)	0.038 45(11)	3.27
6	−0.525(12)	1.2681(6)	0.039 17(16)	0.76
8	−0.526(30)	1.2682(10)	0.039 18(23)	0.85
10	−0.553(55)	1.2688(14)	0.039 31(31)	0.96
12	−0.574(90)	1.2691(18)	0.039 37(40)	1.05
14	−0.47(13)	1.2676(22)	0.039 07(49)	1.17
16	−0.24(23)	1.2649(32)	0.038 51(67)	1.27

also smaller for fixed ξ_{2nd}/L than for fixed U.

Because we had to go to large L_{min} with the simple ansatz (17) we added an analytic correction

$$\bar{\chi} = c + \mathrm{d}L^{2-\eta}. \tag{18}$$

Note also that corrections that decay like L^{-x} with $x \approx 2$ are effectively parametrized by this ansatz. Results for fits with this ansatz are given in table 8 for fixed ξ_{2nd}/L and for fixed U in table 9. We see that a small χ^2/d.o.f. is already reached for $L_{min} = 7$ and 6, respectively. Despite the fact that a χ^2/d.o.f. of the order of one is reached, the results for η do not match within statistical errors. This is a reminder that a small χ^2/d.o.f. does not imply that systematical errors are of the same size as the statistical ones.

Since the statistical error with fixed ξ_{2nd}/L is smaller we take our final result from these fits. In order to estimate systematical errors we compare results of fits with the range L_{min}, L_{max} and $L'_{min} = 2L_{min}$, $L'_{max} = 2L_{max}$. Then the error due to L^{-2} (which we assume to be the leading corrections beyond $L^{-\omega}$) corrections in the second interval should be $\frac{1}{3}$ of the difference of the two results (up to a difference in the distribution of the data with the interval). As our final estimate we take the fit result from $L_{min} = 14$ and $L_{max} = 48$. For comparison we fitted with $L_{min} = 7$ and $L_{max} = 24$. For this interval we get $\eta = 0.038\,00(13)$. Hence the systematical error from L^{-2} corrections should be smaller than 0.000 12 (taking statistical errors into account).

Finally, we checked for systematical errors due to residual leading-order corrections to scaling at $\lambda = 2.0$. For this purpose we fitted our data for $\lambda = 1.0$ and 4.0, also with $L_{min} = 14$ and ansatz (18). We get $\eta = 0.0375(13)$ and 0.0373(13), respectively. Taking into account the statistical errors we find that

$$\left| \frac{\Delta \eta_{eff}}{\Delta c_1(\lambda)} \right| < 0.018. \tag{19}$$

From the previous section we know that the coefficient $c_1(2.0)$ should be smaller than 0.007 (taking the fit result for $L_{min} = 16$ plus the statistical error). Therefore, the systematical error in our final estimate of η due to residual leading-order corrections should be smaller than 0.000 13. As a check we repeated the error analysis using a method similar to [5] and came up with a comparable estimate.

As a final estimate for η we take the result from fitting the magnetic susceptibility at fixed ξ_{2nd}/L with ansatz (18) and $L_{min} = 14$

$$\eta = 0.0381(2)[2]. \tag{20}$$

The estimate of the systematical error is given in the second bracket. It covers residual $L^{-\omega}$ corrections and higher-order corrections.

4.4. The exponent ν

We computed the derivate of the Binder cumulant, U, with respect to κ at the fixed value of the Binder cumulant, $U = 1.243$, and at the fixed value of $\xi_{2nd}/L = 0.5927$. These quantities behave as

$$\overline{\frac{\partial U}{\partial \kappa}} = cL^{1/\nu}. \tag{21}$$

Results of the fits are summarized in tables 10 and 11 for fixed ξ_{2nd}/L and for fixed U, respectively. The χ^2/d.o.f. becomes of the order of one starting from $L_{min} = 8$ and 7, respectively. The statistical errors are slightly smaller in the case of fixed U.

As in the case of the exponent η, in addition to the statistical error we expect systematical errors due to the fact that the coefficient of $L^{-\omega}$ corrections does not vanish exactly and due to sub-leading L^{-2} corrections.

In order to estimate these errors we proceed as in the previous section.

As our final result we take the fit with $L_{min} = 14$ and $L_{max} = 48$ of $\overline{\frac{\partial U}{\partial \kappa}}$ at fixed U. In order to estimate L^{-2} corrections we fitted the data in the interval $L_{min} = 7$ and $L_{max} = 24$. For these lattice sizes we obtain $\nu = 0.6712(2)$. Hence the estimate for a L^{-2} error is $0.0011(5)/3 \approx 0.0005$.

Table 10. Fits of $\overline{\frac{\partial U}{\partial \kappa}}$ at ξ_{2nd}/L fixed with ansatz (21).

L_{min}	$c/2$	ν	χ^2/d.o.f.
6	−0.5542(5)	0.6709(1)	3.82
7	−0.5565(6)	0.6715(2)	1.76
8	−0.5578(7)	0.6719(2)	1.02
10	−0.5586(9)	0.6721(2)	0.96
12	−0.5595(11)	0.6723(3)	0.80
14	−0.5608(13)	0.6727(4)	0.65
16	−0.5620(18)	0.6729(5)	0.69
20	−0.5610(24)	0.6727(6)	0.63
24	−0.5632(36)	0.6732(9)	0.50

Table 11. Fits of $\frac{\partial U}{\partial \kappa}$ at fixed U with ansatz (21).

L_{min}	$c/2$	ν	χ^2/d.o.f.
6	−0.5551(4)	0.6712(1)	2.07
7	−0.5564(5)	0.6716(1)	1.19
8	−0.5572(6)	0.6718(2)	0.80
10	−0.5577(7)	0.6719(2)	0.73
12	−0.5583(9)	0.6721(3)	0.62
14	−0.5593(11)	0.6723(3)	0.51
16	−0.5601(15)	0.6725(4)	0.55
20	−0.5592(21)	0.6723(5)	0.48
24	−0.5610(31)	0.6727(7)	0.30

In order to estimate the error due to residual $L^{-\omega}$ corrections we fitted our data for $\lambda = 1.0$ and 4.0. From $L_{min} = 14$ we obtain $\nu = 0.6706(11)$ for $\lambda = 1.0$ and $\nu = 0.6758(10)$ for $\lambda = 4.0$. Hence

$$\left| \frac{\Delta \nu_{eff}}{\Delta c_1(\lambda)} \right| < 0.04. \tag{22}$$

From the previous section we know that $c_1(2.0) \approx 0.007$. Therefore, the estimate of the systematical error in ν is $0.04 \times 0.007 \approx 0.0003$.

We arrive at our final estimate

$$\nu = 0.6723(3)[8] \tag{23}$$

where the statistical error is given in the first bracket and the systematical error that is given in the second bracket covers L^{-2} and residual $L^{-\omega}$ corrections.

5. Comparison with the literature

In table 12 we give, for comparison, recent results for critical exponents. Critical exponents for the XY-universality class were calculated using the high-temperature series expansions, the ϵ-expansion, perturbation theory in three dimension and Monte Carlo simulations.

The Monte Carlo simulations [14–17] are performed for the two-component XY model, which is the $\lambda = \infty$ limit of the model discussed in the present paper. The three-component XY model is studied in [18]. In this model the field variable is a three-component unit vector and the coupling of the third component vanishes. All these Monte Carlo studies use a simple cubic lattice. The high-temperature series expansion of [20] is performed for the two-component XY model on the simple cubic and on the body-centred cubic lattice.

Our result for ν is consistent, within error bars, with (almost) all other theoretical results given in table 12. The result of the Monte Carlo study [16] seems to be a little too small in magnitude. Our error bar is smaller than that of all previous estimates. Our estimate for η is consistent with the other theoretical estimates except with some of the Monte Carlo results. The values of [15, 16] are too small compared with our present estimate. Note that in these studies no careful check of systematical errors due to corrections to scaling was performed. On the other hand, the result of [14], which takes into account $L^{-\omega}$ corrections, is larger than our result by two standard deviations.

In contrast to the one-component case [6] our result for the correction to scaling exponent ω is consistent with that obtained with field theoretic methods.

Table 12. Recent results for critical exponents obtained with Monte Carlo simulations (MC), ϵ-expansion, perturbation theory in three dimensions (3D, PT) and high-temperature (HT) series expansions. When only ν and γ are given in the reference we computed η with the scaling law. These cases are indicated by *. In [1] a result for α and in [17] a result for α/ν are given. In the table these results are converted to ν using the scaling relation $\alpha = 2 - d\nu$. For a discussion see the text.

Reference	Method	ν	η	ω
Present work	MC	0.672 3(3)[8]	0.0381(2)[2]	0.79(2)
[15]	MC	0.670(2)	0.025(7)*	
[16]	MC	0.662(7)	0.026(6)	
[17]	MC	0.672 4(17)		
[14]	MC	0.672 1(13)	0.042(2)	
[18]	MC	0.669 3(58)	0.035(5)	
[19]	3D, PT	0.670 3(15)	0.0354(25)	0.789(11)
[19]	ϵ, bc	0.668 0(35)	0.0380(50)	0.802(18)
[19]	ϵ, free	0.671	0.0370	0.802(18)
[20]	HT	0.674(2)	0.039(7)*	
[1]	^{4}He	0.670 95(13)		
[2]	^{4}He	0.670 5(6)		
[3]	^{4}He	0.670 8(4)		

Experimental results for the exponent ν have been obtained for the λ-transition of ^{4}He. These results have smaller error bars than our Monte Carlo result. The experimental results are all smaller then our value but the error bars still touch.

6. Conclusion and outlook

In this paper we have considerably improved the accuracy of the theoretical estimate of ν of the 3D XY universality class. In particular, in addition to the statistical error we give a careful estimate of systematical errors that are caused by corrections to scaling. Our value, $\nu = 0.6723(3)[8]$, is consistent with other theoretical estimates. However, it is larger than the experimental results obtained from the λ-transition of ^{4}He [1–3] that give values from 0.6704 to 0.6709 with an error in the last digit. It would be interesting to further improve the theoretical estimate to the claimed accuracy of the experimental results. This could be achieved by simulating at our best estimate for $\lambda_{opt} = 2.1$ and using linear lattice sizes roughly twice as large as in the present study to reduce the effect of sub-leading corrections. At a sustained statistical accuracy this would require about 10 years of CPU-time on a modern PC.

In addition to critical exponents amplitude ratios are universal and have been experimentally determined for the λ-transition of ^{4}He. For example, the specific heat behaves, in the neighbourhood of the phase transition, as

$$C = A_{\pm}|t|^{-\alpha}(1 + D_{\pm}|t|^{\theta} + E_{\pm}t) + B \tag{24}$$

where $t = (T - T_c)/T_c$ is the reduced temperature. The constants $A_{\pm}$, $D_{\pm}$, $E_{\pm}$ and B depend on the system that is considered. The subscript $\pm$ indicates the low- and high-temperature phase. However, renormalization group predicts the ratio A_+/A_- to be universal. Setting $\lambda = \lambda_{opt}$ leads to $D_{\pm} = 0$ which greatly simplifies the determination of A_+/A_- in a Monte Carlo simulation. For a Monte Carlo determination of A_+/A_- based on the XY model see [17].

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